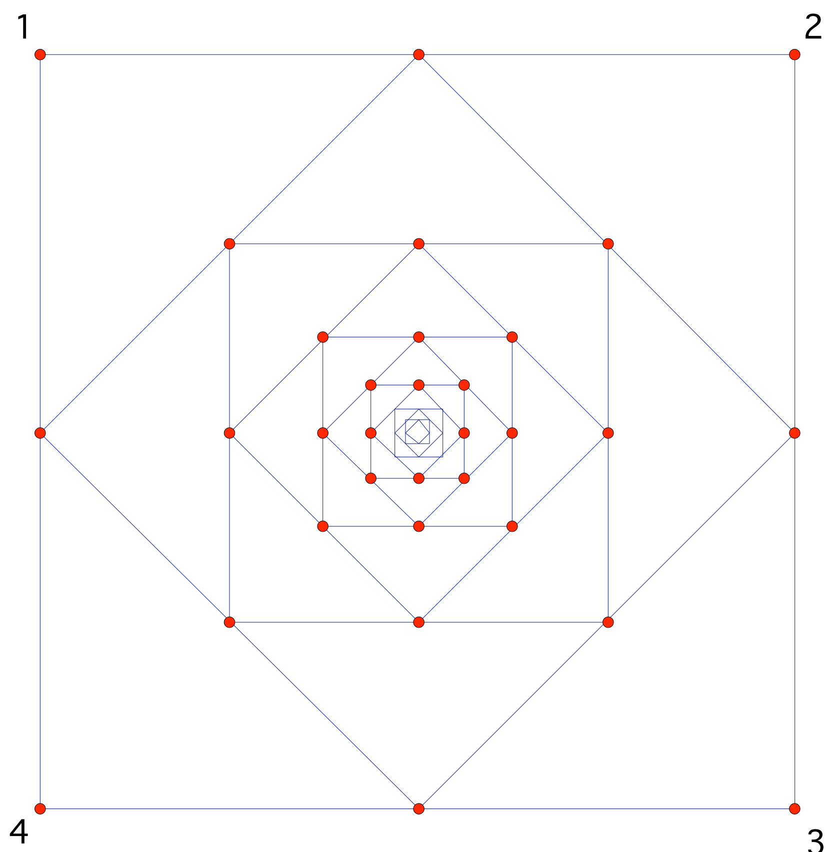
Julia Robinson Mathematics Festival

Directed By Joshua Zucker

Difference Engine

The numbers 1, 2, 3, and 4 are written at the corners of a large square. At each step, at the midpoint of each side, write the positive (or absolute value of the) difference between the numbers at its ends, so in the diagram below, you would write the numbers 1, 1, 1, and 3 at the midpoints of the sides, forming a new square.



Then repeat that process, writing the positive difference between the endpoints at the midpoint of each side. Eventually, things will get boring…

1. Make a sensible definition of "boring" and determine how many steps it takes for this process to become boring.

2. Using only integers in the range 0 through 1, what's the longest it can be until this process becomes boring?

3. How about if you can use integers in the range 0 through 2?

4. 0 through 10? 0 through 100?

5. Prove that the process always becomes boring.

6. What if you start with 1, 2, π, 4 instead of 1, 2, 3, 4?

7. How about $1, \sqrt{2}, \pi, \frac{14}{3}$?

8. Does this change your proof? That is, did you assume in your proof that the numbers were all integers? Does the process still always become boring even if the numbers aren't integers?

Now, what if we use other shapes besides a square, say, an *n*-sided regular polygon?

9. When $n = 1$ or 2, if you can even call that a polygon, things get boring really fast. Explain why.

10. When $n = 3$, you may need to change your definition of boring. Why? Do all sequences become boring in the same way?

11. In general, which values of *n* can use the same definition of boring as the square?

12. Can you establish a general upper limit on how long it will take for a given set of numbers to become boring, based on the largest number in the set and the number of sides of the *n*-gon?