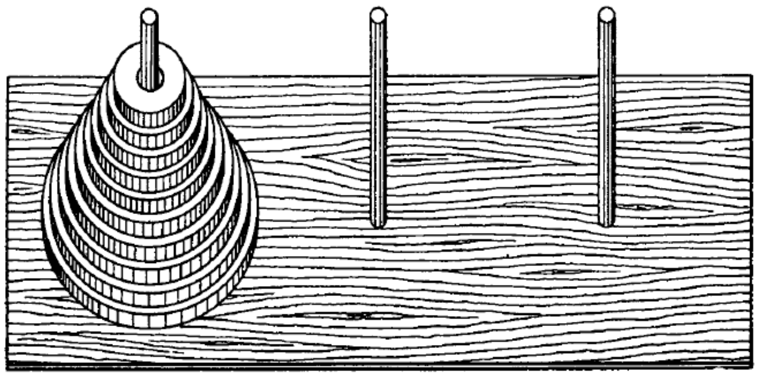
Julia Robinson Mathematics Festival

Directed By Joshua Zucker

The Tower of Hanoi - and Beyond

Legend has it that in Hanoi there is a tower of 64 disks of different sizes, initially all stacked on peg A as shown, and a group of monks working tirelessly to move the disks from peg A to peg B. Only one disk at a time may be moved, and at all times only a smaller disk may ever be on top of a larger (never a larger on top of a smaller). When the monks complete their task, the legend says the world will end.



1. Perhaps this is too easy, but how many moves will it take to complete the moving of the tower if there is only one disk?
2. How many moves will it take if there are two disks?
3. How many moves will it take for three disks?
4. How long will it take for four disks? Generalize. How many moves will it take for *n* disks?
5. If the monks never make a mistake, and can move one disk every second, 24 hours per day, how many years will it take for them to complete their tower of 64 disks?

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6. Now repeat the previous problems, but with an additional rule: every disk must be moved either to or from peg C, never directly between pegs A and B.

7. Show that, in solving the previous exercise, you must encounter every legal arrangement of disks on the three pegs.

8. Let Q_n be the number of moves required to move a tower of n disks from peg A to peg B with all moves being "forward" (that is, A to B, or B to C, or C to A) and similarly R_n the number of moves required to move a tower of n disks from peg B to peg A with all moves being "forward". Prove that

$$Q_n = \begin{cases} 0, & \text{if } n = 0 \\ 2R_{n-1}, & \text{if } n > 0 \end{cases} \qquad R_n = \begin{cases} 0, & \text{if } n = 0 \\ Q_n + Q_{n-1} + 1, & \text{if } n > 0. \end{cases}$$

9. A Double Tower of Hanoi contains $2n$ disks of n different sizes, two of each size. How many moves does it take to move a double tower from peg A to peg B (under the usual one-disk-at-a-time rules of the initial problem)?

10. Generalize: what if you have a given number of repeats of each peg, such as 1 of the smallest, 2 of the next smallest, 3 of the next smallest, and 5 of the biggest disk?